ON CALCULATION OF CHAMFER DISTANCE AND LIPSCHITZ COVERS IN DIGITAL IMAGES

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Abstract. We study the chamfer distance transformations of binary digital images and corresponding Lipschitz covers of grayscale images. Validity of the double scan algorithm in arbitrary dimension is proved.

Keywords: chamfer distance, Lipschitz cover, top-hat procedure, background elimination

1 Introduction

Distance transformation is an operation with a lot of applications in image processing and in analysis of spatial patterns. Special distance transformations, e.g. chamfer or city block distance, can be calculated especially easily by a sequential double scan algorithm [3,9]. The same algorithm can be used for the calculation of the lower Lipschitz cover of a grayscale image [7] that is equivalent to a grayscale opening by a cone [4]. The Lipschitz cover can be applied for the elimination of a slowly varying image background by subtraction of the lower Lipschitz cover (a top-hat procedure).

The distance is usually defined as a symmetric function that is positive for distinct points. If we relax these assumptions we obtain a function called quasi-distance that can be useful to model *e.g.* directional positional relations in images [1], or real situations like uphill and downhill paths. We will show that the quasi-distance can also be calculated by the double scan algorithm. Moreover, the notion of quasi-distance can simplify description and validation of the double scan algorithm that inherently contains the measurements of the quasi-distance to the preceding image elements during the scan.

2 Chamfer quasi-distance in digital images

The elements of the n-dimensional digital image are arranged in a regular lattice and they can be indexed by $\mathbf{Z}^n$.

The digital image is a function $f : X \to \mathbf{R} \cup \{-\infty, +\infty\}$, where $X \subset \mathbf{Z}^n$. The binary image is a digital image with range in $\{0,1\}$.

Definition 2.1: Chamfer mask M is a digital image with a finite domain $D(M)$ such that $M \geq 0$ and $M(\mathbf{0}) = 0$.

The path from x to y, $x, y \in X$ is a sequence of vectors $v_1, v_2, .., v_m \in D(M)$ such that

$$x - y = \sum_{k=1}^{m} v_k \ .$$

Chamfer quasi-distance between $x, y \in X$, $X \subset \mathbf{Z}^n$, is

$$d_M(x,y) = \inf_{\{v\}} \sum_{k=1}^{m} M(v_k), \qquad (1)$$

where sequence of vectors $v_1, v_2, .., v_m \in D(M)$ is path v from x to y.

The chamfer quasi-distance, defined in (1) fulfills $d_M(x,x) = 0$, $d_M(x,y) \geq 0$, $d_M(x,y) + d_M(y,z) \geq d(x,z)$. Related chamfer quasi-norm $n_M(x) = d_M(x,0)$ is positively homogeneous $n_M(ax) = an_M(x)$ for $a \in \mathbf{Z}$, $a \geq 0$.

The notion of quasi-distance is more general than notion of distance:

a) d is symmetric, i.e. $d_M(x,y) = d_M(y,x)$ iff M is symmetric: $M(x) = M(-x)$.

b) $d_M(x,y) = 0$ does not imply $x \neq y$ iff there is $x \neq 0$ such that $M(x) = 0$.

c) $d_M(x,y)$ attains infinite value for x, y that can not be connected by vectors from $D(M)$.

Definition 2.2: Let $x, y \in Z^n$. Then $x <_L y$ lexicographically iff there is $0 \leq j \leq n$ such that $x_j < y_j$ and $x_k = y_k$ for $k > j$. Let us define the two intervals bounded by **0** from one side $\mathbf{Z}^{n-} = \{x \in \mathbf{Z}^n, x <_L \mathbf{0}\}$, $\mathbf{Z}^{n+} = \{x \in \mathbf{Z}^n, x >_L \mathbf{0}\}$.

Definition 2.3: Let f be a function and let d be a quasi-distance on $D(f)$, then f is d-Lipschitz function with respect to d iff for every x, y from $D(f)$

$$f(x) - f(y) \leq d(x,y). \qquad (2)$$

Let f be a function. Then the greatest d-Lipschitz function $f(x) - f(y) \leq d(x,y)$ is called the lower d-Lipschitz cover of *f*. Lipschitz condition (2) represents continuity in discrete spaces.

3 Chamfer distance transformation algorithms in 2D

The distance transformation converts a binary digital image into a gray-level image with pixels having value of the distance to the nearest feature. It can be achieved using only local operations of a small neighborhood of a pixel. The chamfer distance transformation of a binary image can be computed either by a sequential algorithm or by a parallel algorithm.

In next section we assume $X \subset \mathbf{Z}^2$. Then we can denote a pixel of the image as $a_{i,j} = f(i,j)$. Computing algorithms for the chamfer distance transformation in arbitrary dimensions are very similar and you can find them in [2].

The parallel algorithm uses a parallel chamfer mask (Fig.1) and is defined by the recurrent relation

$$\begin{aligned} a_{i,j}^0 &= 0 && \text{if } a_{i,j} = 0, \\ a_{i,j}^0 &= m + n && \text{if } a_{i,j} = 1, \qquad (3)\\ a_{i,j}^{k+1} &= \inf\{a_{i+u_1,j+u_2}^k + M(u) \mid u \in D(M), (i+u_1, j+u_2) \in D(f)\}, \end{aligned}$$

where u is a vector, $u = (u_1, u_2)$, m is the width of the image, n is the height of the image, $i \in \{1,..,m\}, j \in \{1,..,n\}$.

Algorithm stops on fixpoint, it means if $a_{i,j}^{k+1} = a_{i,j}^k$ for all *i,j*.

We used in (3) the value $m+n$ that represents the infinity in the algorithm, because any real distance in the image is less than $m+n$.

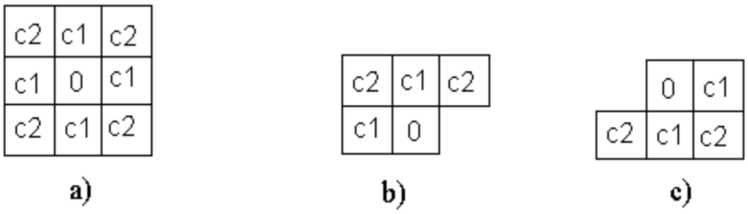

Fig. 1: 3 x 3 chamfer masks: a) parallel chamfer M, b) forward sequential M^-*,*
c) backward sequential M^+*.*

The sequential algorithm presented in [8] has two steps: forward and backward scan. Both of these scans use its own chamfer masks M^- and M^+ (Fig.1), with the signs in the sense of lexicographical order, defined in **2.2**. The coefficients c_1 and c_2 represent the gradient step in horizontal and vertical directions and in diagonal direction, respectively. Usually $c_1 = 1, c_2 = \sqrt{2}$ or if we require that these values are natural numbers, $c_1 = 2, c_2 = 3$. A more precise calculation also with larger chamfer masks is possible to find in [6].

The first forward scan (with output matrix $(a^1_{i,j})$) begins from the upper left corner of the image and scans the rows all over the image to the bottom right corner. The second backward scan (with output matrix ($a^2_{i,j}$)) has the opposite direction of the scanning from the bottom right corner to the upper left corner:

$$a^1_{i,j} = 0 \qquad \text{if } a_{i,j} = 0,$$

$$a^1_{i,j} = \inf\{m+n\} \cup \{a^1_{i+u_1,j+u_2} + M^-(u) \mid u \in D(M^-) \setminus \{0\}, (i+u_1, j+u_2) \in D(f)\}$$

$$\text{if } a_{i,j} = 1,$$

$$a^2_{i,j} = \inf\{a^1_{i,j}\} \cup \{a^2_{i+u_1,j+u_2} + M^+(u) \mid u \in D(M^+) \setminus \{0\}, (i+u_1, j+u_2) \in D(f)\}.$$

In the computation of the value of $a^1_{i,j}$ we use the value of $a^1_{i+u_1,j+u_2}$, but we already have these values from previous computation for all $u \in D(M^+)$, because $u <_L \mathbf{0}$. Analogous situation is in the second backward scan.

This algorithm can be easily extended for the calculation of the lower d-Lipschitz cover of a grayscale image [7]:

$$a^1_{i,j} = \inf\{a_{i,j}\} \cup \{a^1_{i+u_1,j+u_2} + M^-(u) \cdot slope \mid u \in D(M^+) \setminus \{0\}, (i+u_1, j+u_2) \in D(f)\},$$

$$a^2_{i,j} = \inf\{a^1_{i,j}\} \cup \{a^2_{i+u_1,j+u_2} + M^+(u) \cdot slope \mid u \in D(M^+) \setminus \{0\}, (i+u_1, j+u_2) \in D(f)\},$$

where *slope* is a parameter controlling gradient in the resulting image.

A similar algorithm computes the upper d-Lipschitz cover:

$$a^1_{i,j} = \sup\{a_{i,j}\} \cup \{a^1_{i+u_1,j+u_2} - M^-(u) \cdot slope \mid u \in D(M^-) \setminus \{0\}, (i+u_1, j+u_2) \in D(f)\},$$

$$a^2_{i,j} = \sup\{a^1_{i,j}\} \cup \{a^2_{i+u_1,j+u_2} - M^+(u) \cdot slope \mid u \in D(M^+) \setminus \{0\}, (i+u_1, j+u_2) \in D(f)\}.$$

The Lipschitz cover is a very useful tool for the elimination of a slowly varying image background. There are some examples in the figures (Fig.2, Fig.3).

In 24-bit (or 32-bit) colored pictures the same algorithm is used for every color channel.

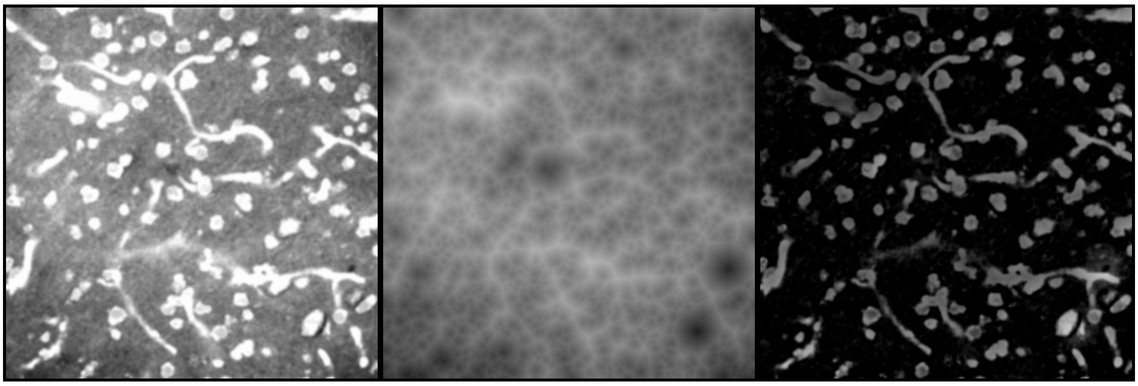

Fig. 2: An original microscopic image (on the left), the Lipschitz lower cover (in the middle) and the image after using a Lipschitz top-hat filter (on the right).

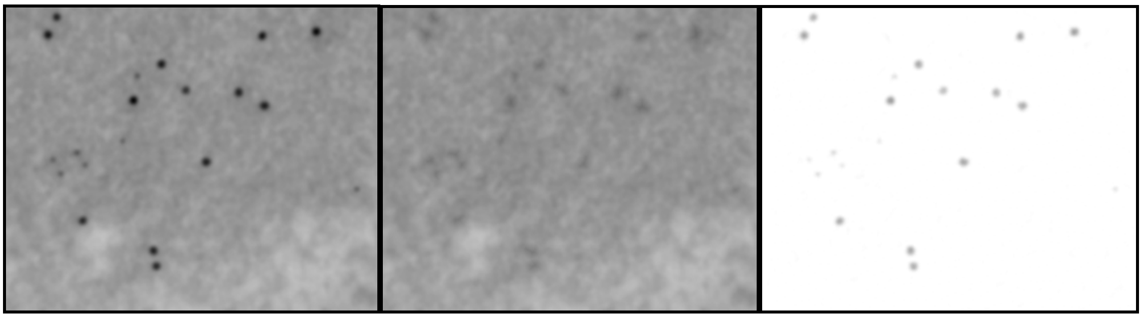

Fig. 3: An original electron-microscopic image of immunogold labels (on the left), the Lipschitz lower cover (in the middle) and the image after using a Lipschitz top-hat filter (on the right).

4 Chamfer distance in digital image of arbitrary dimension

Now we will assume that $X \subset \mathbf{Z}^n$, n > 0. It can be proved that the lower d-Lipschitz cover of f is

$$g(x) = \inf_{y \in D(f)} \left(f(y) + d(x,y) \right). \tag{4}$$

The sequential algorithm for the distance transformation scans the image elements in the lexicographic order calculating the lower d_{M^+}-Lipschitz cover h of the image f taking the minimum of $h(x+y)+M^+(y)$ for $x+y \in D(f)$, $y \in D(M^+)$, $y <_L \mathbf{0}$ and of $f(x)$ in the first step and then scans the image elements in the anti-lexicographic order calculating lower d_{M^-}-Lipschitz cover g of the image h, taking minimum of $g(x+y)+M^-(y)$ for $x+y \in D(f)$, $y \in D(M^+)$, $y <_L \mathbf{0}$ and of $h(x)$ in the second step.

According to the following theorem the result of the double scan algorithm is the lower chamfer quasi-distance Lipschitz cover of the digital image.

Theorem 4.1 : Let f be an image and M a chamfer mask. Let M^+ and M^- be chamfer masks such that $M^+(x) = M(x)$ if $x <_L \mathbf{0}$ and $M^+(x)$ is not defined if $x >_L \mathbf{0}$, $M^-(x) = M(x)$ if $x >_L \mathbf{0}$ and $M^-(x)$ is not defined if $x <_L \mathbf{0}$. A lower d_M-Lipschitz cover of f is a lower d_{M^-}-Lipschitz cover of a lower d_{M^+}-Lipschitz cover of f.

Proof: Let g be a lower d_M-Lipschitz cover of f, then $g(x) = \inf_y (f(y) + d_M(x,y))$. The path $\{v_k\}$ (4) minimizing $d_M(x,y) = \inf_{\{v_k\}} \sum_{k=1}^{m} M(v_k)$ can be partitioned between $\mathbf{Z}^{n+}$ and $\mathbf{Z}^{n-}$:

$$d_M(x,y) = \sum_{k=1}^{m} M^+(v_k) + \sum_{k=1}^{m} M^-(v_k)$$

and the statement of the theorem follows from

$$g(x) = \inf_z \left(\inf_y (f(y) + d_{M^+}(z,y)) + d_{M^-}(x,z) \right).$$

Proposition 4.2: Let d be a quasi-distance. The distance transformation of a binary digital image b is an image $DT_{b,d}$ such that $D(DT_{b,d}) = D(b)$,

$$DT_{b,d}(x) = \inf_{b(y)=0} d(x,y).$$

It is easy to see that $DT_{b,d}$ is a lower d-Lipschitz cover of the function $g \circ b$, $g(0) = 0$, $g(1) = 1$. Then by the theorem 4.1 the double scan algorithm calculates also the chamfer quasi-distance transformation of the binary image.

5 Conclusion

The chamfer distance relatively well approximates the Euclidean distance and is widely used because of its relatively small computational requirements as it imposes only 2 scans of the n-dimensional image independently of the dimension of the image. A more precise approximation [5] requires 2^n scans in n-dimensional image, hence for $n > 2$ the algorithms for Euclidean distance transformation that are separable in dimension and require n-scans are preferable [4].

Acknowledgement

The study was supported by the Academy of Sciences of the Czech Republic (Grants A100110502, A500200510 and AV0Z 50110509).

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